# Piotr Lewczuk
# A Plugboard Connections Problem

## The Problem

Consider an electric plugboard with even number $n$ of sockets and a set of $n/2$ cables ("connectors"), each of which can connect two sockets (generating "a connection" or "a pair"). For convenience, we denote the sockets with letters: A, B, C, etc, hence a connection is simply denoted by two letters (like AB, where AB and BA is the same, of course). An example of a board with $n = 8$ sockets and 4 connectors is presented in Figure 1, with four pairs: AC, BF, DE, and GH. Now, assuming that all sockets are connected, how many configurations of connections are possible? Speaking other words, how many distinct subsets of size $n/2$ of pairs of elements can be generated from a set of size $n$?

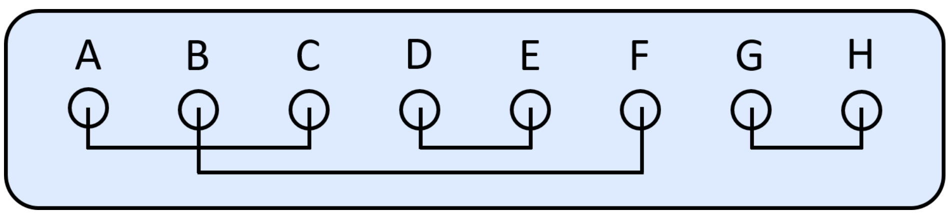


Figure 1: A board with eight sockets and four connectors

For example, for a small board with 4 sockets and 2 connectors, there are three possibilities:

$$\{AB, CD\}$$
$$\{AC, BD\}$$
$$\{AD, BC\}.$$

Enumerating all possibilities for a board with 6 sockets is still plausible; it is easy to show there are 15 of them. But what about larger boards with, say, dozens or hundreds of sockets?

## 1st Solution

The first approach to solve the problem comes from combinatorics: the first pair can be generated in

$$\binom{n}{2}$$

different ways. The second pair (after excluding two sockets from the pool) can be generated in

$$\binom{n-2}{2}$$

ways, and so one. Combining all binomial coefficients leads to the following number of *sequences*:

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$$\binom{n}{2}\cdot\binom{n-2}{2}\cdot\binom{n-4}{2}\cdot\ldots\cdot\binom{2}{2}$$

$$=\frac{n!}{2(n-2)!}\cdot\frac{(n-2)!}{2(n-4)!}\cdot\frac{(n-4)!}{2(n-6)!}\cdot\ldots\cdot\frac{2}{2\cdot 0!}$$

$$=\frac{n!}{2^{\frac{n}{2}}}.$$

This is the number of all possible sequences where the order, in which we generate the pairs, matters (intuitively, remember we said: "the first pair", "the second pair", etc.). Observe that when $n/2-1$ pairs are built, the last pair is determined, which is reflected by the last factor, $\binom{2}{2}=1$. If our board were an old fashioned analogue phone center operator board, the number above would tell us something about the order, in time, in which people made phone connections until all phones became busy (but without considering who called whom: AB and BA is the same pair or the same connection). But hence we are not interested in the order of making connections, we need to consider the number of possible re-orderings of a sequence of $n/2$ elements (pairs of sockets), which is $(n/2)!$. Hence, the answer for our problem becomes:

$$\frac{n!}{2^{\frac{n}{2}}(\frac{n}{2})!}. \tag{1}$$

A quick check: for $n=4$ and $n=6$, we obtain 3 and 15 configurations, respectively, as expected. And this could be the end of the story, if we did not consider a different approach to the problem, leading to an interesting observation.

## 2nd Solution

Remember, all sockets must be connected. Hence, we take the socket A and connect it with an arbitrary socket. Obviously, there are $n-1$ possible connections. Next, we take the first free socket (perhaps B, if not used in the first pair) and connect it with an arbitrary free socket. There are $n-3$ pairs possible. An so on. When all sockets are connected, we have

$$(n-1)\cdot(n-3)\cdot(n-5)\cdot\ldots\cdot 3\cdot 1 \tag{2}$$

possible configurations of the connections. Again, a quick check (for $n=4$ and $n=6$) shows we are good. Now, observe that the last expression is the product of all positive odd numbers less than an even number $n$. So, interestingly, as a by-product we have developed an expression for the product of a sequence of odd numbers. We summarize this observation in a theorem and prove it:

*(1) Product of all positive odd numbers less than any even number $n$ equals Eq. (1).*
*(2) Product of all positive even numbers less than or equal to any even number $n$ equals*

$$2^{\frac{n}{2}}\left(\frac{n}{2}\right)!. \tag{3}$$

The proof of the first part follows by induction. First, we re-state the inductive hypothesis as

$$\frac{n!}{(\frac{n}{2})!\prod_{i=1}^{n/2}(2i-1)}=2^{\frac{n}{2}}, \tag{4}$$

where $\prod_{i=1}^{n/2}(2i-1)$ is the product of the positive odd numbers less than $n$. The base case is for the lowest positive even number, $n = 2$, and it is trivial to show that Eq. (4) holds true. Assuming that the inductive hypothesis holds for $n$, for the inductive case, $m = n + 2$, we have:

$$\begin{aligned}
&\frac{m!}{(\frac{m}{2})!\prod_{i=1}^{m/2}(2i-1)}\\
=&\frac{(n+2)!}{(\frac{n+2}{2})!\prod_{i=1}^{n/2+1}(2i-1)}\\
=&\frac{n!(n+1)(n+2)}{(\frac{n}{2})!\frac{n+2}{2}(n+1)\prod_{i=1}^{n/2}(2i-1)}\\
=&\frac{n!}{(\frac{n}{2})!\prod_{i=1}^{n/2}(2i-1)}\cdot\frac{2(n+1)(n+2)}{(n+2)(n+1)}\\
=&2^{\frac{n}{2}}\cdot 2\\
=&2^{\frac{n+2}{2}}\\
=&2^{\frac{m}{2}}.
\end{aligned}$$

As for the second part, obviously the product of all positive even numbers less than or equal to $n$ results from dividing the product of all natural numbers less than or equal to $n$ (which is $n!$) by the product of odd numbers, which we just proved to be given in Eq. (1); this immediately leads to Eq. (3). □

## A Generalization

What about a problem with $k \leq \frac{n}{2}$ connectors? Say, not all neighbours around are willing to make a phone call or our board has limited connecting capacity. The problem can be rephrased as that we have a subset of $2k \leq n$ sockets which are all connected and a subset of the remaining sockets which are not. Of course, for $2k = n$ we obtain the special case discussed above. Observe that $n$ does not have to be even in this setting; it can be odd as long as $2k < n$.

The problem can be solved in two steps. First, we observe that the number of subsets of size $2k$ (subsets of sockets which are all connected) that can be generated from a set of size $n$ is

$$\binom{n}{2k},$$

and for the second step, we use the previously developed expression, taking into account we have now a "sub-board" with $2k$ sockets, all connected. This gives

$$\begin{aligned}
&\frac{n!}{(2k)!(n-2k)!}\cdot\frac{(2k)!}{2^k k!}\\
=&\frac{n!}{2^k(n-2k)!k!} \qquad (5)
\end{aligned}$$

possible configurations.

## Not Only Plugboards

The beauty of mathematics relies, among other things, upon that many expressions are applicable (or generalisable) to problems which do not seem to have anything in common when considered superficially (say, on a level how they physically look like), but turn out to be governed by the same logic as soon as we dig deeper to the underlying mathematics.

Our small problem is not an exception. Consider blackjack, the most popular casino card game. There are usually three to eight people (and the dealer) in the game and each player initially gets exactly two cards from a standard 52-card deck (very often more than one deck is used, but let's keep things simple). Now, observe that to find the number of the initial configurations of the cards dealt in blackjack, we apply the same rule which we developed for the plugboard connections, using $n = 52$ for the number of cards in a deck (which correspond to the sockets of a board) and $k$ for the number of players (who may be seen as "connectors"). By dealing two cards to a player we "connect" them as we do with the sockets of a plugboard. Here we assume, in absolute contrast to the gambling reality, that it does not matter who obtained which pair of cards; it only matters which pairs were dealt[1].

Another - seemingly unrelated - example. Consider a group of 20 tourists seeking accommodation in a hotel offering 10 single rooms and 5 double ones. Assuming there are no particular preferences who wants to stay in a single room and who in a double room, and who wants to stay with whom in a double room, the question is, how many configurations of placement of tourists in the two room types are possible? Is there any analogy to the plugboard problem? If so, what are $n$ and $k$ numbers here[2]?

## The German Enigma

Finally, if you think our puzzle is an abstract game just for fun, let me tell you it is (or was) not! It has a - literally - painful dimension as practical as a question of life and death has. A German Enigma, perhaps one of the most elegant cipher machines built ever and heavily used during World War II, had, among other nasty mechanical devices, a plugboard (*Steckerbrett)* with 26 pairs of sockets, one pair for each letter of the alphabet, and 10 connecting cables (in its Army and Air Force version, used from October 1936 on). It did not look exactly like the board in this puzzle but the idea was the same: each letter could be bidirectionally connected to another one. Now, for $n = 26$ and $k = 10$, we obtain from our Eq. (5)

$$150,738,274,937,250$$

configurations, which is exactly the number given by Enigma researchers[3]. This is a huge number, contributing significantly to the complexity of Enigma. To put it in perspective, if we set up one configuration every second, it would take almost five million years to check all of them.

1 If we insist that it does matter who obtained which pair, Eq. (5) needs to be simply multiplied by $k$!

2 $n = 20$ for the number of tourists and $k = 5$ for the number of double rooms, which are analogous to the "connectors".

3 An exhaustive analysis of mathematics of Enigma, provided by a National Security Agency expert, you will find in: A. Ray Miller, *The Cryptographic Mathematics of Enigma*, NSA, 2019 (see particularly table on page 9); other sources citing this number are: www.codesandciphers.org.uk/enigma/steckercount.htm or www.cryptomuseum.com/crypto/enigma/i/sb.htm

## A Small Simulation in Python

The script below simulates $M$ plugboards, each with $n$ sockets and $k$ connectors. It returns the number of distinct configurations and one exemplary configuration as a $k$-elements list of paired sockets. To obtain correct results for boards larger than $n = 12$, large number of simulations need to be performed (say, $M \geq 10^6$), which may take a while, otherwise the results may be underestimated. You may compare the number obtained with the simulation with theoretical number obtained from Eq. (5).

```
########################################################################################
from random import sample as rs

class Board:
    def __init__(self, n, k):
        self.sockets = [i for i in range(n)]
        self.configuration = []
    def set_config(self):
        for p in range(k):
            c = rs(self.sockets, 2)
            c.sort()
            self.configuration.append(c)
            self.sockets = [i for i in self.sockets if i not in c]
        self.configuration.sort(key = lambda c: c[0])

n = 10
k = 4
M = 50000

list_of_configurations = []
for m in range(M):
    b = Board(n, k)
    b.set_config()
    if b.configuration not in list_of_configurations:
        list_of_configurations.append(b.configuration)

print(len(list_of_configurations),
      ["".join([chr(c[0]+65), chr(c[1]+65)]) for c in b.configuration])

## 4725 ['AI', 'BG', 'CF', 'EH']
########################################################################################
```